ON THE ACTION OF CONCENTRATED FORCES AND MOMENTS ON AN ELASTIC THIN SHELL OF ARBITRARY SHAPE

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In this article the nature of the peculiarities (in the neighborhood of a point of the application of concentrated forces and moments) of the functions u, ..., T_1, ..., H_1 which represent the displacements, stresses, and moments in the shell are investigated. The problem is solved in its general setting (the shell is of arbitrary shape) for general equations of moments of the linear theory of shells. Besides, for shells of positive Gaussian curvature, the problem is considered also for the equations of the membrane ("momentless") theory of shells. The results obtained in this work are generalizations of results obtained earlier for shells of particular type. For example, the case when the concentrated forces and moments act on a spherical shell was treated in an article by Gol'denveizer [1]; the case of a cylindrical shell was considered by Darevskii [2]. Chernykh [3] dealt with shells of arbitrary shape, but did not carry out the investigation to the end.

For the solution of the problem the author makes use of results obtained for fundamental solutions of partial differential equations of the elliptic type. The information required is presented, for example, in the works of Gel'fond and Shilov [4], Levi [5], Ion [6], Lopatinskii [7] and others.

The problem is the following: to express in explicit form the main singularities of the solutions u, v, w of the differential equations of equilibrium for a shell when this shell is subjected to the action of a concentrated force and concentrated moment.

A fundamental solution of a differential equation $L(\Phi) = 0$ is a solution of the operator equation $L(\Phi) = \delta(\xi - \xi_0)$, where L is a differential operator; $\xi = \{\xi_1, \ldots, \xi_n\}$, $\xi_0 = \{\xi_{10}, \ldots, \xi_{n0}\}$ are vectors in an n-dimensional space, and δ is the Dirac function.

The connection between the problem and the fundamental solutions becomes evident if one recalls that a concentrated force is usually considered as the limit of a definite distribution of the loading, or as the solution of a differential equation having a definite singularity. In what follows we shall use the first definition with certain refinements which permit us to apply the theory of generalized functions.

By a concentrated force, applied at the point $\xi = 0$, we shall mean the limit of the sequence of distributed loads q_ν satisfying the following conditions.

1. For every given $M > 0$, and for $|a| \leqslant M$, $|b| \leqslant M$, the quantities

$$\left| \int_a^b q_\nu(\xi)\, d\xi \right|$$

are bounded by a constant independent of a, b, and ν (depending on M).

2. For all a and b different from zero

$$\lim_{\nu\to\infty} \int_a^b q_\nu(\xi)\, d\xi = \begin{cases} 0 & (a < b < 0,\ 0 < a < b) \\ 1 & (a < 0 < b) \end{cases}$$

The limit of functions q_ν possessing these properties is called the Dirac δ-function in the theory of generalized functions [4]. Hence the accepted definition of a concentrated force is equivalent to that of a Dirac δ-function. The sequence of the functions q_ν will be called a δ-type sequence.

We shall next give the definition of a concentrated moment. A concentrated moment is the limit as $\nu \to \infty$ of the distributed loadings q_ν, which have the form of the function represented in Fig. 1.

The branches of the function q_ν to the right and left of $\xi = 0$, form δ-type sequences. We shall require that as $\nu \to \infty$ these loadings yield a constant moment of intensity one with respect to the point $\xi = 0$, and that the resultant be zero.

The first of these requirements yields the equation

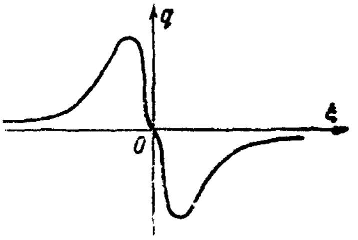
$$\lim_{\nu\to\infty} \int_a^b \xi q_\nu(\xi)\, d\xi = 1 \qquad (a < 0 < b) \tag{1}$$

The second requirement leads to the following equation

$$\lim_{\nu\to\infty}\int_a^b q(\xi)\,d\xi = 0 \quad (a < 0 < b) \tag{2}$$

From (1) one obtains, through integration by parts, the result

$$\lim_{\nu\to\infty}\int_a^b d\xi\int_a^\xi q_\nu(\eta)\,d\eta = -1 \tag{3}$$



With the aid of (2) and (3) we can show that

$$\lim_{\nu\to\infty} q_\nu = -\delta'(0)$$

Here δ' denotes the derivative of the δ-function which, according to [4], is defined in the following way.

Let $\varphi(\xi)$ be any function belonging to the class of k-times $(k \geqslant 2)$ differentiable functions. If $f(\xi)$ is such that

$$\int_c^d f(\xi)\,\varphi(\xi)\,d\xi = -\varphi'(\xi_0) \qquad (c < \xi_0 < d)$$

then $f(\xi) = \delta'(\xi - \xi_0)$. Thus we must show that

$$\lim_{\nu\to\infty}\int_a^b q_\nu(\xi)\,\varphi(\xi)\,d\xi = \varphi'(0) \tag{4}$$

Indeed, integrating (4) by parts, we obtain

$$\lim_{\nu\to\infty}\int_a^b q_\nu(\xi)\,\varphi(\xi)\,d\xi = \lim_{\nu\to\infty}\varphi\int q_\nu(\xi)\,d\xi\,\Big|_a^b - \lim_{\nu\to\infty}\int_a^b \varphi'(\xi)\,d\xi\int_a^\xi q_\nu(\eta)\,d\eta$$

The first term on the right-hand side is equal to zero by (2), the second one is equal to $\varphi'(0)$ in view of (3) and the arbitrariness of a and b.

Let the given shell be represented by means of orthogonal coordinates (α, β). The intensity of a unit concentrated force, and the intensity of unit concentrated moments directed along the α and β axes, can be expressed by means of the functions

$$\frac{\delta}{AB}, \qquad \frac{1}{AB^2}\frac{\partial\delta}{\partial\beta}, \qquad -\frac{1}{A^2B}\frac{\partial\delta}{\partial\alpha}$$

respectively, where A and B are the coefficients of the first quadratic form of the surface of the shell. In what follows, we shall assume that the surface is expressed in terms of orthogonal, conjugate coordinates.

We shall begin with the differential equations of equilibrium and with displacements which can be expressed in the form

$$\begin{aligned} \Delta_{11}u + \Delta_{12}v + \Delta_{13}w &= -\frac{1-\sigma^2}{2Eh}X \\ \Delta_{21}u + \Delta_{22}v + \Delta_{23}w &= -\frac{1-\sigma^2}{2Eh}Y \\ \Delta_{31}u + \Delta_{32}v + \Delta_{33}w &= \frac{1-\sigma^2}{2Eh}Z \\ \Delta_{ik} &= \overset{\circ}{\Delta}_{ik} + \Delta_{ik} \end{aligned} \tag{5}$$

Here u, v, and w are displacements, X, Y, and Z are the components of the loading, Δ_{ik}° are operators containing derivatives of higher order, and the Δ_{ik}' are operators involving the remaining terms.

The content of the operators Δ_{ik} will be taken from the equations of equilibrium given by Gol'denveizer [9]. Then the matrix from the operators Δ_{ik}° will have the form

$$\begin{Vmatrix} p_1\left(D_{\alpha\alpha}^{\,2} + \frac{1-\sigma}{2}D_{\beta\beta}^{\,2}\right) & q_1 D_{\alpha\beta}^{\,2} & \frac{h^2}{3R_1}D_\alpha\Delta \\ q_2 D_{\alpha\beta}^{\,2} & p_2\left(\frac{1-\sigma}{2}D_{\alpha\alpha}^{\,2} + D_{\beta\beta}^{\,2}\right) & \frac{h^2}{3R_2}D_\beta\Delta \\ \frac{h^2}{3}\left[\frac{1}{R_1}D_\alpha\Delta + \frac{1-\sigma}{2}pD_{\alpha\beta\beta}^{\,3}\right] & \frac{h^2}{3}\left[\frac{1}{R_2}D_\beta\Delta - \frac{1-\sigma}{2}pD_{\alpha\alpha\beta}^{\,3}\right] & \frac{h^2}{3}\Delta^2 \end{Vmatrix}$$

Here (6)

$$p_i = 1 + \frac{h^2}{3R_i^{\,2}}, \qquad q_i = \frac{1+\sigma}{2} + \frac{h^2}{3R_1R_2} - \frac{1-\sigma}{2}\frac{h^2}{3R_i^{\,2}} \qquad (i = 1, 2)$$

$$p = \frac{1}{R_1} - \frac{1}{R_2}, \qquad \Delta = \frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2} + \frac{1}{B^2}\frac{\partial^2}{\partial\beta^2}, \qquad p_3 = 1 + \frac{h^2}{3R_1R_2}$$

Further, we have introduced the symbols

$$D_\alpha = \frac{1}{A}\frac{\partial}{\partial\alpha}, \qquad D_{\alpha\alpha}^{\,2} = \frac{1}{A^2}\frac{\partial^2}{\partial\alpha^2}, \qquad D_\beta = \frac{1}{B}\frac{\partial}{\partial\beta}, \ldots$$

which will be used in the sequel. R_1 and R_2 are the principal radii of curvature; $2h$ is the thickness of the shell. It is assumed that A, B, R_1 and R_2 are nonvanishing functions which have derivatives of the required order, and that A and $B \neq \infty$.

Let us denote by l_{ik} the algebraic cofactor of the term Δ_{ik}° in the matrix $\|\Delta_{ik}^{\circ}\|$, and let us write down the matrix $\|l_{ik}\|$

$$\left\| \begin{matrix} \frac{h^2}{3}\left(\frac{1-\sigma}{2} D_{\alpha\alpha}^{2} + D_{\beta\beta}^{2}\right)\Delta^2, & -\frac{h^2}{3}\frac{1+\sigma}{2} D_{\alpha\beta}^{2}\Delta^2, & \frac{h^2}{3} D_\alpha\left[-\frac{1-\sigma}{2R_1} D_{\alpha\alpha}^{2} + r_{13} D_{\beta\beta}^{2}\right]\Delta \\ -\frac{h^2}{3}\frac{1+\sigma}{2} D_{\alpha\beta}^{2}\Delta^2, & \frac{h^2}{3}\left(D_{\alpha\alpha}^{2} + \frac{1-\sigma}{2} D_{\beta\beta}^{2}\right)\Delta^2, & \frac{h^2}{3} D_\beta\left[r_{23} D_{\alpha\alpha}^{2} - \frac{1-\sigma}{2R_2} D_{\beta\beta}^{2}\right]\Delta \\ \frac{h^2}{3} D_\alpha\left[-\frac{1-\sigma}{2R_1} D_{\alpha\alpha}^{2} + r_{31} D_{\beta\beta}^{2}\right]\Delta, & \frac{h^2}{3} D_\beta\left[r_{32} D_{\alpha\alpha}^{2} - \frac{1-\sigma}{2R_2} D_{\beta\beta}^{2}\right]\Delta^2, & \frac{1-\sigma}{2}\Delta^2 \end{matrix} \right\|$$

where

$$r_{13} = \frac{1}{R_2} - \frac{3-\sigma}{2R_1}, \qquad r_{23} = \frac{1}{R_1} - \frac{3-\sigma}{2R_2}$$
$$r_{31} = \frac{1+\sigma}{2R_2} - \frac{1}{R_1}, \qquad r_{32} = \frac{1+\sigma}{2R_1} - \frac{1}{R_2}$$

The system of equilibrium equations is a system of the elliptic type, and the elliptic operator Λ has the form

$$\Lambda = |\Delta_{ik}^{\circ}| = \frac{1-\sigma}{2}\frac{h^2}{3}\left(p_2 D_{\alpha\alpha\alpha\alpha}^{4} + 2p_3 D_{\alpha\alpha\beta\beta}^{4} + p_1 D_{\beta\beta\beta\beta}^{4}\right)\Delta^2$$

In what follows we shall drop the quantities $h^2/3R_iR_k$ $(i,\ k = 1,\ 2)$ because they are small compared to one. We shall assume that

$$\Lambda = \frac{1-\sigma}{2}\frac{h^2}{3}\Delta^4 \tag{7}$$

Let us investigate the problem for the case when the concentrated force acts in the direction parallel to a coordinate axis. Then, on the right-hand sides of the system (5), one must set $X = \delta/AB$, $Y = Z = 0$, and one has to find the solution of the system

$$\Delta_{11}u + \Delta_{12}v + \Delta_{13}w = -\frac{1-\sigma^2}{2Eh}\frac{\delta}{AB}$$
$$\Delta_{21}u + \Delta_{22}v + \Delta_{23}w = 0$$
$$\Delta_{31}u + \Delta_{32}v + \Delta_{33}w = 0$$

Making use of Levi's method [5], we express u, v and w in the form

$$u = l_{11}\Phi(\alpha, \beta, \alpha_0, \beta_0) + \iint\limits_{(G)} l_{11}\Phi(\alpha, \beta, \xi, \eta) f_1(\xi, \eta, \alpha_0, \beta_0)\, d\xi d\eta$$
$$v = l_{12}\Phi(\alpha, \beta, \alpha_0, \beta_0) + \iint\limits_{(G)} l_{12}\Phi(\alpha, \beta, \xi, \eta) f_2(\xi, \eta, \alpha_0, \beta_0)\, d\xi d\eta \tag{8}$$

$$w = l_{13}\Phi(\alpha, \beta, \alpha_0, \beta_0) + \iint\limits_{(G)} l_{13}\Phi(\alpha, \beta, \xi, \eta) f_3(\xi, \eta, \alpha_0, \beta_0)\, d\xi d\eta$$

where f_i is still an unknown function, $\Phi(\alpha, \beta, \alpha_0, \beta_0)$ is a fundamental solution of the equation

$$-\frac{2Eh}{1-\sigma^2} AB\Lambda\Phi = \delta(\alpha_0, \beta_0)$$

Levi gives a general method for finding Φ. For the case under consideration when Λ has the form (7), the principal part of the fundamental solution ψ, that is the part of highest singularity, has the form

$$\psi = -\frac{3}{36 \times 64 \times 2\pi h^3 (1+\sigma)} r^6 \ln r^2, \qquad r^2 = A^2(\alpha - \alpha_0)^2 + B^2(\beta - \beta_0)^2$$

We shall express the function Φ in the form $\Phi = \psi + \Psi$. Here, and in what follows, Ψ will represent all terms of lower singularities than those of the terms written out explicitly. The symbol Ψ may thus stand for different expressions in different formulas. We give certain relations which we shall need

$$\Delta\psi = -\frac{\varkappa}{4 \times 64\pi} r^4 \ln r^2 + \Psi, \qquad \Delta^2\psi = -\frac{\varkappa}{32\pi} r^2 \ln r^2 + \Psi$$

$$D_{\alpha\alpha}^2 \Delta\psi = -\frac{\varkappa}{64\pi} r^2 \ln r^2 - \frac{\varkappa}{32\pi} A^2 (\alpha - \alpha_0)^2 \ln r^2 + \Psi$$

$$D_{\alpha\alpha}^2 \Delta^2\psi = -\frac{\varkappa}{8\pi} \ln r^2 + \Psi, \qquad \varkappa = \frac{6}{h^3(1+\sigma)}$$

The derivatives of $\Delta\psi$, $\Delta^2\psi$ with respect to β can be written down in a similar way. For the determination of the unknown function f_i one can obtain a system of Fredholm integral equations of the second kind by substituting (8) into the original equations. However, here we shall not consider the construction of the functions f_i because the aim of our investigation is the finding of the principal singularities, which are contained in the first terms of the right-hand sides of the equations (8). We shall prove this last assertion.

In order to find the principal singularities of the solutions, it is sufficient to consider a system of equations in which only the highest order derivatives are retained (see, e.g. [6]). In the case under consideration, if the original system of equations should have constant coefficients, then the expressions $u = l_{11}\Phi$, $v = l_{12}\Phi$, and $w = l_{13}\Phi$ would give the required solutions.

The solutions of a system with variable coefficients will be of the same form in the neighborhood of the point where the concentrated forces are applied. This follows from the work of Bers [8]. There it is proved that the singularity of the fundamental solution of a differential equation with variable coefficients in the neighborhood of a singular point consists of the singularities of the fundamental solution of the equation with constant coefficients, and of singularities of lower order than the principal ones. This assertion is valid also for the derivatives of the fundamental solution.

The problem of the action of concentrated forces Y and Z is solved in an analogous manner. The results of the computation are given in Table 1.

Let us now consider the action of a concentrated moment. Having defined it as the limit of a normally distributed loading, we shall look for the solution of the following system

$$\begin{aligned}
&\triangle_{11}u + \triangle_{12}v + \triangle_{13}w = 0 && \left(M_1 = \frac{1}{AB} D_\beta \delta\right) \\
&\triangle_{21}u + \triangle_{22}v + \triangle_{23}w = 0 && \left(M_2 = -\frac{1}{AB} D_\alpha \delta\right) \\
&\triangle_{31}u + \triangle_{32}v + \triangle_{33}w = \frac{1-\sigma^2}{2Eh} M_i && (i = 1, 2)
\end{aligned} \tag{9}$$

where M_1 stands for the moment about the α-axis, while M_2 represents the moment about the β-axis.

We solve the system (9) by the same method, except the function Φ in (8) is replaced by Φ_1 which is a solution of the equation

$$\Lambda\Phi_1 = \frac{1-\sigma^2}{2Eh} M_i$$

From the theory of generalized functions it is known that if Φ is a fundamental solution of the equation $\Lambda\Phi = \delta$, the $\partial\Phi/\partial\alpha$ is a solution of the equation $\Lambda\Phi = \partial\delta/\partial\alpha$. From this it follows, as was to be expected, that the principal parts of the solutions in the given case can be obtained by simple differentiations of the obtained principal parts of the solutions u_z, v_z and w_z for the action of the concentrated force Z with respect to the same variable with respect to which the δ-function is differentiated that stands on the right-hand side of the third equation of (9). The solution is thus obtained easily. For example, if the action is that of the concentrated moment directed along the α-axis, then the solution has the form $u = D_\beta u_z, \ \ldots, \ w = D_\beta w_z, \ T_1 = D_\beta T_{1z}, \ \ldots, \ H_1 = D_\beta H_{1z}$, where $T_1, \ \ldots, \ H_1$ are the stresses and moments in the shell.

For the moment along the β-axis we obtain $u = -D_\alpha u_z, \ldots, H_1 = -D_\alpha H_{1z}$.

We shall next consider the membrane equations of a shell of positive Gaussian curvature (the last restriction is essential, because the differential equations will be of the elliptic type only for such shells).

Let us split the matrix of the operators of the system Δ_{ik} into the principal and secondary matrices in the same way in which this was done for the moment equations $\Delta_{ik} = \Delta_{ik}^{\circ} + \Delta_{ik}'$. In accordance with the differential equations of equilibrium in the displacements [9], the matrix has the form

$$\|\mathring{\Delta}_{ik}\| = \left\| \begin{matrix} D_{\alpha\alpha}^2 + \frac{1-\sigma}{2} D_{\beta\beta}^2 & \frac{1+\sigma}{2} D_{\alpha\beta}^2 & -\left(\frac{1}{R_1} + \frac{\sigma}{R_2}\right) D_\alpha \\ \frac{1+\sigma}{2} D_{\alpha\beta}^2 & \frac{1-\sigma}{2} D_{\alpha\alpha}^2 + D_{\beta\beta}^2 & -\left(\frac{\sigma}{R_1} + \frac{1}{R_2}\right) D_\beta \\ -\left(\frac{1}{R_1} + \frac{\sigma}{R_2}\right) D_\alpha & -\left(\frac{\sigma}{R_1} + \frac{1}{R_2}\right) D_\beta & 2 r_{11} \end{matrix} \right\|$$

The matrix of the algebraic cofactors l_{ik} will be symmetric in the given case, namely, $l_{ik} = l_{ki}$, $i, k = 1, 2$ and its elements have the form

$$l_{11} = (1-\sigma)\left(r_{11} D_{\alpha\alpha}^2 + \frac{1+\sigma}{R_1^2} D_{\beta\beta}^2\right), \qquad l_{12} = -\frac{1-\sigma}{2} p^2 D_{\alpha\beta}^2$$

$$l_{13} = \frac{1-\sigma}{2} D_\alpha \left[\left(\frac{1}{R_1} + \frac{\sigma}{R_2}\right) D_{\alpha\alpha}^2 + r_{13} D_{\beta\beta}^2\right]$$

$$l_{22} = (1-\sigma)\left(\frac{1+\sigma}{R_2^2} D_{\alpha\alpha}^2 + r_{11} D_{\beta\beta}^2\right)$$

$$l_{23} = \frac{1-\sigma}{2} D_\beta \left[r_{23} D_{\alpha\alpha}^2 + \left(\frac{1}{R_2} + \frac{\sigma}{R_1}\right) D_{\beta\beta}^2\right], \qquad l_{33} = \frac{1-\sigma}{2} \Delta^2$$

Here

$$r_{11} = \frac{1}{2}\left(\frac{1}{R_1^2} + \frac{2\sigma}{R_1 R_2} + \frac{1}{R_2^2}\right), \qquad r_{13} = \frac{2+\sigma}{R_1} - \frac{1}{R_2}, \qquad r_{23} = \frac{2+\sigma}{R_2} - \frac{1}{R_1}$$

The operator $\Lambda = |\Delta_{ik}^{\circ}|$ has the form

$$\Lambda = \frac{(1-\sigma)(1-\sigma^2)}{2} \left(\frac{1}{R_2} D_{\alpha\alpha}^2 + \frac{1}{R_1} D_{\beta\beta}^2\right)^2$$

TABLE 1.

	X	Y	Z
u	$-\varkappa_1 \ln r^2$	$\varkappa_2 \frac{xy}{r^2}$	$\varkappa_3 D_\alpha\, [m_{13}^{(1)} - (1+\sigma)\, py^2/r^2]\, r^2 \ln r^2$
v	$\varkappa_2 \frac{xy}{r^2}$	$-\varkappa_1 \ln r^2$	$\varkappa_3 D_\beta\, [m_{23}^{(1)} + (1+\sigma)\, px^2/r^2]\, r^2 \ln r^2$
w	$-\varkappa_3 D_\alpha\, [m_{31}^{(1)} - 2py^2/r^2]\, r^2 \ln r^2$	$-\varkappa_3 D_\beta\, [m_{32}^{(1)} + 2px^2/r^2]\, r^2 \ln r^2$	$-\varkappa_4 r^2 \ln r^2$

Here

$$\varkappa_1 = \frac{(3-\sigma)(1+\sigma)}{16\pi Eh}, \qquad \varkappa_2 = \frac{(1+\sigma)^2}{8\pi Eh}, \qquad \varkappa_3 = \frac{1+\sigma}{64\pi Eh}, \qquad \varkappa_4 = \frac{3(1-\sigma^2)}{32\pi Eh^3}$$

$$p = \frac{1}{R_1} - \frac{1}{R_2}, \qquad m_{13}^{(1)} = \frac{1}{2}\left(\frac{1+\sigma}{R_2} - \frac{5-3\sigma}{R_1}\right), \qquad m_{23}^{(1)} = \frac{1}{2}\left(\frac{1+\sigma}{R_1} - \frac{5-3\sigma}{R_2}\right)$$

$$m_{31}^{(1)} = \frac{1}{R_2} - \frac{3-2\sigma}{R_2}, \qquad m_{32}^{(1)} = \frac{1}{R_1} - \frac{3-2\sigma}{R_2}$$

$$x = A(\alpha - \alpha_0), \qquad y = B(\beta - \beta_0), \qquad r = \sqrt{A^2(\alpha-\alpha_0)^2 + B^2(\beta-\beta_0)^2}$$

TABLE 2.

	X	Y	Z
T_1	$-\frac{1}{8\pi} D_\alpha\left[(3+\sigma)\ln r^2 + 2(1+\sigma)\frac{y^2}{r^2}\right]$	$\frac{1}{8\pi} D_\beta\left[(1-\sigma)\ln r^2 - 2(1+\sigma)\frac{y^2}{r^2}\right]$	$\frac{1}{4\pi}\left[m_{13}^{(2)}\ln r^2 - 2t - (1+\sigma)p\frac{x^2y^2}{r^4}\right]$
T_2	$\frac{1}{8\pi} D_\alpha\left[(1-\sigma)\ln r^2 - 2(1+\sigma)\frac{x^2}{r^2}\right]$	$-\frac{1}{8\pi} D_\beta\left[(3+\sigma)\ln r^2 + 2(1+\sigma)\frac{x^2}{r^2}\right]$	$\frac{1}{4\pi}\left[m_{23}^{(2)}\ln r^2 + 2t + (1+\sigma)p\frac{x^2y^2}{r^4}\right]$
S_1	$-\frac{1}{4\pi} D_\beta\left[(1-\sigma)\ln r^2 + 2(1+\sigma)\frac{y^2}{r^2}\right]$	$-\frac{1}{4\pi} D_\alpha\left[(1-\sigma)\ln r^2 + 2(1+\sigma)\frac{x^2}{r^2}\right]$	$\frac{1}{8\pi} D_\alpha{}^2{}_\beta\left[m_{33}^{(2)} + (1+\sigma)t\right] r^2\ln r^2$
G_1	$\frac{h^2}{24\pi} D_\alpha{}^3{}_{\beta\beta}\left[m_{41}^{(2)} + 2t\right] r^2\ln r^2$	$\frac{h^2}{24\pi} D_\alpha{}^3{}_{\alpha\beta}\left[m_{42}^{(2)} + 2t\right] r^2\ln r^2$	$\frac{1}{4\pi}\left[(1+\sigma)\ln r^2 + 2(1-\sigma)\frac{x^2}{r^2}\right]$
G_2	$\frac{h^2}{24\pi} D_\alpha{}^3{}_{\beta\beta}\left[m_{51}^{(2)} - 2t\right] r^2\ln r^2$	$\frac{h^2}{24\pi} D_\alpha{}^3{}_{\alpha\beta}\left[m_{52}^{(2)} - 2t\right] r^2\ln r^2$	$\frac{1}{2\pi}\left[(1+\sigma)\ln r^2 + 2(1-\sigma)\frac{y^2}{r^2}\right]$
H_1	$-\frac{h^2}{6\pi} D_\beta\left[m_{61}^{(2)}\ln r^2 - \frac{1+\sigma}{R_2}\frac{x^2}{r^2} + p\frac{x^2y^2}{r^4} + \frac{2}{R_1}\frac{y^2}{r^2}\right]$	$-\frac{h^2}{6\pi} D_\alpha\left[m_{62}^{(2)}\ln r^2 - \frac{1+\sigma}{R_1}\frac{y^2}{r^2} - p\frac{x^2y^2}{r^4} + \frac{2}{R_2}\frac{x^2}{r^2}\right]$	$\frac{1-\sigma}{2\pi}\frac{xy}{r^2}$

In Table 2

$$t = \frac{x^2}{R_2} + \frac{y^2}{R_1}, \qquad m_{13}^{(2)} = \frac{1}{4}\left(\frac{5+\sigma}{R_1} - \frac{1-3\sigma}{R_2}\right)$$

$$m_{23}^{(2)} = \frac{1}{4}\left(\frac{5+\sigma}{R_2} - \frac{1-3\sigma}{R_1}\right), \qquad m_{33}^{(2)} = \frac{1-3\sigma}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

$$m_{41}^{(2)} = \frac{1+2\sigma}{R_2} + \frac{1-2\sigma}{R_1}, \qquad m_{42}^{(2)} = -\frac{1}{R_1} - \frac{5}{R_2}, \qquad m_{51}^{(2)} = \frac{1}{R_2} - \frac{3}{R_1}$$

$$m_{52}^{(2)} = \frac{3+2\sigma}{R_1} + \frac{3-2\sigma}{R_2}, \qquad m_{61}^{(2)} = \frac{3}{R_1} - \frac{2(1+2\sigma)}{R_2}, \qquad m_{62}^{(2)} = \frac{3}{R_2} - \frac{2(1+2\sigma)}{R_1}$$

TABLE 3.

	X	Y	Z
U	$-\varkappa\, m_{11}^{(3)} \ln r_1^2$	$\varkappa p^2 \frac{x_1 y_1}{r_1^2}$	$-\frac{\varkappa}{2} D_\alpha \left[(2(1+\sigma) - m_{13}^{(3)}) \ln r_1^2 - m_{23}^{(3)} \frac{y_1^2}{r_1^2}\right]$
V	$\varkappa p^2 \frac{x_1 y_1}{r_1^2}$	$-\varkappa\, m_{22}^{(3)} \ln r_1^2$	$-\frac{\varkappa}{2} D_\beta \left[(2(1+\sigma) + m_{13}^{(3)}) \ln r_1^2 - m_{23}^{(3)} \frac{x_1^2}{r_1^2}\right]$
W	$-\frac{\varkappa}{2} D_\alpha \left[\left(2(1+\sigma) - m_{13}^{(3)}\right) \ln r_1^2 - m_{23}^{(3)} \frac{y_1^2}{r_1^2}\right]$	$-\frac{\varkappa}{2} D_\beta \left[(2(1+\sigma) + m_{13}^{(3)}) \ln r_1^2 - m_{23}^{(3)} \frac{x_1^2}{r_1^2}\right]$	$\frac{\varkappa}{2} \Delta^2 r_1^2 \ln r_1^2$

Here

$$x_1 = \sqrt{R_2} A (\alpha - \alpha_0), \quad y_1 = \sqrt{R_1} B (\beta - \beta_0), \quad r_1 = \sqrt{R_2 A^2 (\alpha - \alpha_0)^2 + R_1 B (\beta - \beta_0)^2}$$

$$m_{11}^{(3)} = \frac{R_2}{R_1^2} + \frac{1}{R_2} + \frac{2+4\sigma}{R_1}, \qquad m_{22}^{(3)} = \frac{R_1}{R_2^2} + \frac{1}{R_1} + \frac{2+4\sigma}{R_2}$$

$$\varkappa = \frac{\sqrt{R_1 R_2}}{16\pi E h}, \qquad m_{13}^{(3)} = \frac{R_1^2 - R_2^2}{R_1 R_2}, \qquad m_{23}^{(3)} = 2\frac{(R_1 - R_2)^2}{R_1 R_2}$$

TABLE 4.

	X	Y	Z
T_1	$\frac{1}{4\pi}\sqrt{\frac{R_1}{R_2}}D_\alpha \ln r_1^2$	$-\frac{1}{4\pi}\sqrt{\frac{R_1}{R_2}}D_\beta \ln r_1^2$	$-\frac{\sqrt{R_1R_2}}{4\pi}D_{\alpha\alpha}^2 \ln r_1^2$
T_2	$-\frac{1}{4\pi}\sqrt{\frac{R_2}{R_1}}D_\alpha \ln r_1^2$	$\frac{1}{4\pi}\sqrt{\frac{R_2}{R_1}}D_\beta \ln r_1^2$	$-\frac{\sqrt{R_1R_2}}{4\pi}D_{\beta\beta}^2 \ln r_1^2$
S_1	$\frac{1}{4\pi}\sqrt{\frac{R_2}{R_1}}D_\beta \ln r_1^2$	$\frac{1}{4\pi}\sqrt{\frac{R_1}{R_2}}D_\alpha \ln r_1^2$	$\frac{\sqrt{R_1R_2}}{4\pi}D_{\alpha\beta}^2 \ln r_1^2$

Here

$$G_1 = -\frac{2Eh}{3(1-\sigma^2)}(D_{\alpha\alpha}^2 + \sigma D_{\beta\beta}^2)w_i, \qquad G_2 = -\frac{2Eh}{3(1-\sigma^2)}(\sigma D_{\alpha\alpha}^2 + D_{\beta\beta}^2)w_i$$

$$H_1 = \frac{2Eh^3}{3(1+\sigma)}D_{\alpha\beta}^2 w_i \qquad (i = X,\ Y,\ Z)$$

Note. The index i indicates that w must be taken from Table 3 for the corresponding force.

Here R_1 and R_2 are of the same sign, and, therefore Λ is an elliptic operator.

The method of solution is here repeated in the same order as for the moment equations. Its description is, therefore, omitted. Only a few expressions will be given which are needed for better understanding.

The function ψ, which denotes the principal part of the fundamental solution of the equation

$$\Lambda\Phi = \frac{1-\sigma^2}{2Eh}\frac{\delta}{AB}$$

for the membrane operator Λ, has the form

$$\psi = \frac{\sqrt{R_1R_2}}{16\pi Eh(1-\sigma)}r_1^2 \ln r_1^2, \qquad r_1^2 = A^2R_2(\alpha-\alpha_0)^2 + B^2R_1(\beta-\beta_0)^2$$

Performing computations analogous to the preceding ones, we obtain the principal singularities of the functions u, v and w (Table 3).

The components of deformation $\varepsilon_1, \ldots, \tau$ are found by differentiating the functions u, v and w. The stresses and moments $T_1, \ldots, H_1$ are expressed in terms of $\varepsilon_1, \ldots, \tau$ with the aid of the relations of elasticity. Performing the calculations, which will not be given here, we obtain the results which are given in Tables 2 to 4 for the general as well as for the membrane cases.

The results of this work for a circular cylindrical shell have been compared with the results obtained by Darevskii [2]. It was found that the asymptotic formulas given here for u, v, T_1, T_2, S_1 and S_2, under the action of the concentrated forces X and Y, and also G_1 and G_2, under the action of the force Z, coincide with the corresponding formulas in [2]. The remaining formulas do not agree. One can show that the disagreement is caused by the fact that in the present work and in [2] there are used different conditions of elasticity. The proposed method is applicable for arbitrary versions of the elasticity relations.

For a spherical shell the results of this article coincide with the results of Gol'denveizer [1].

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